

Generalized matrix models and AGT correspondence at all genera

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ABSTRACT

We study generalized matrix models corresponding to n -point Virasoro conformal blocks on Riemann surfaces with arbitrary genus g . Upon AGT correspondence, these describe four dimensional $\mathcal{N} = 2$ $SU(2)^{n+3g-3}$ gauge theories with generalized quiver diagrams. We obtain the generalized matrix models from the perturbative evaluation of the Liouville correlation functions and verify the consistency of the description with respect to degenerations of the Riemann surface. Moreover, we derive the Seiberg-Witten curve for the $\mathcal{N} = 2$ gauge theory as the spectral curve of the generalized matrix model, thus providing a check of AGT correspondence at all genera.

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1 Introduction

The distinctive feature of M-theory is the description in geometrical terms of non-perturbative phases of superstrings. This approach is very effective for local geometries, where the dynamics of gravitational degrees of freedom gets decoupled and we gain a framework for the description of non-perturbative gauge theory dynamics. M-theory beautifully encodes the Seiberg-Witten geometry of four dimensional $\mathcal{N} = 2$ theories in terms of M5-brane compactifications [1, 2, 3]. In particular in [3] a full class of generalized quiver gauge theories has been described in terms of multiple M5-brane systems covering a generic punctured Riemann surface $\mathcal{C}_{g,n}$. For example, for $\mathcal{C}_{0,n}$ and $\mathcal{C}_{1,n}$ one recovers Witten's constructions of linear and circular quivers in the appropriate degeneration limits.

In this context a very intriguing relation between the partition function of four dimensional $SU(2)^{n+3g-3}$ superconformal $\mathcal{N} = 2$ gauge theories [4] and Liouville theory on $\mathcal{C}_{g,n}$ has been discovered in [5]. This proposal has been a subject of intensive investigations and refinements from different viewpoints. Evidence for this conjecture as well as complete proofs for some cases can be found in [6, 7]. Extensions to higher rank gauge groups and Toda field theories were introduced and discussed in [8]. The refinement of the correspondence in presence of gauge theory observables has been presented and studied in [9, 10, 11]. Moreover, some arguments for the derivation of the AGT correspondence were proposed in the M-theory context in [12] and via

matrix models in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

Here we would like to address this correspondence from a complementary point of view, explaining how to recover the geometry of the M-theory set-up and the Seiberg-Witten data in the wildest generality. To this end we derive a generalized matrix model from Liouville theory on $\mathcal{C}_{g,n}$ and study its large N limit recovering the gauge theory Seiberg-Witten curve as its spectral curve. This provides a check of the AGT conjecture at all genera.

In Section 2 we derive the generalized matrix model – as extended Selberg integrals – starting from the Coulomb gas representation of the residues of the perturbative Liouville theory correlators. The matrix model potential that we get has the form anticipated by [13] and in the elliptic case it coincides with the one derived in [20].

In Section 3 we discuss the stability of this picture and its consistency with respect to the degeneration of the curve $\mathcal{C}_{g,n}$ in general and present the degenerations of punctured tori as an explicative example.

In Section 4 we analyze the large N limit and show how, by consistently adapting to our case the standard matrix model techniques, one gets a spectral curve in terms of quadratic differentials on $\mathcal{C}_{g,n}$ precisely reproducing the Seiberg-Witten curve and differential proposed in [3].

We leave our concluding remarks to Section 5 and devote an Appendix to the detailed study of the degenerations of the $\mathcal{C}_{2,0}$ Seiberg-Witten data.

2 From Liouville theory to generalized matrix model

In this section we derive the generalized matrix model which corresponds to the n point conformal block on a Riemann surface \mathcal{C}_g of genus g . We derive it from the perturbative calculation of the correlation function of the Liouville theory by following the discussion in [23].

The n -point function of the Liouville theory on \mathcal{C}_g is given by the following path integral

$$A \equiv \left\langle \prod_{k=1}^n e^{-2m_k \phi(w_k, \bar{w}_k)} \right\rangle_{\text{Liouville on } \mathcal{C}_g} \equiv \int \mathcal{D}\phi(z, \bar{z}) e^{-S[\phi]} \prod_{k=1}^n e^{-2m_k \phi(w_k, \bar{w}_k)}, \quad (2.1)$$

where the Liouville action is given by

$$S[\phi] = \frac{1}{4\pi} \int d^2z \sqrt{g} (g^{ab} \partial_a \phi \partial_b \phi + QR\phi + 4\pi\mu e^{2b\phi}). \quad (2.2)$$

We divide the Liouville field into the zero mode and the fluctuation $\phi(z, \bar{z}) = \phi_0 + \tilde{\phi}(z, \bar{z})$, obtain-

ing

$$A = \int \mathcal{D}\tilde{\phi} e^{-\tilde{S}} \prod_{k=1}^n e^{-2m_k \tilde{\phi}(w_k, \bar{w}_k)} \int_{-\infty}^{+\infty} d\phi_0 e^{-\mu e^{2b\phi_0} \int d^2z \sqrt{g} e^{2b\tilde{\phi}} e^{-\frac{Q\phi_0}{4\pi} \int d^2z \sqrt{g} R} e^{-2\phi_0 \sum_k m_k}}, \quad (2.3)$$

where

$$\tilde{S} = \frac{1}{4\pi} \int d^2z \sqrt{g} (g^{ab} \partial_a \tilde{\phi} \partial_b \tilde{\phi} + Q R \tilde{\phi}). \quad (2.4)$$

We can integrate out the zero mode ϕ_0 as

$$\int_{-\infty}^{+\infty} d\phi_0 e^{-\mu e^{2b\phi_0} \int d^2z \sqrt{g} e^{2b\tilde{\phi}} e^{-2(g-1)Q\phi_0} e^{-2\phi_0 \sum_k m_k}} = \frac{\mu^N \Gamma(-N)}{2b} \left(\int d^2z \sqrt{g} e^{2b\tilde{\phi}} \right)^N, \quad (2.5)$$

where we have used $\int d^2z \sqrt{g} R = 4\pi\chi = 8\pi(1-g)$ and N is defined as

$$N \equiv \frac{1}{b} \sum_k m_k + \frac{Q}{b} (1-g). \quad (2.6)$$

Therefore, the n -point function can be written as

$$A = \frac{\mu^N \Gamma(-N)}{2b} \int \mathcal{D}\tilde{\phi} e^{-\tilde{S}} \left(\int d^2z \sqrt{g} e^{2b\tilde{\phi}} \right)^N \prod_{k=1}^n e^{-2m_k \tilde{\phi}(w_k, \bar{w}_k)}. \quad (2.7)$$

When $N \in \mathbb{Z}_{\geq 0}$, the correlator diverges due to the factor $\Gamma(-N)$. The residues A_N at these simple poles are computed then in perturbation theory in b around the free scalar field action (2.4). From now on, our convention is that

$$\langle \dots \rangle_{\text{free on } \mathcal{C}_g} = \int \mathcal{D}\tilde{\phi} e^{-\frac{1}{4\pi} \int d^2z \sqrt{g} g^{ab} \partial_a \tilde{\phi} \partial_b \tilde{\phi}} \dots, \quad (2.8)$$

which leads to

$$A_N = \frac{(-\mu)^N}{2bN!} \left\langle e^{-\frac{Q}{4\pi} \int d^2z \sqrt{g} R \tilde{\phi}(z)} \prod_{i=1}^N \int d^2z_i \sqrt{g} e^{2b\tilde{\phi}(z_i)} \prod_{k=1}^n e^{-2m_k \tilde{\phi}(w_k)} \right\rangle_{\text{free on } \mathcal{C}_g}. \quad (2.9)$$

The condition (2.6) ensures momentum conservation in the free theory.

Here we choose as a reference volume form $d^2z \sqrt{g} = |\omega(z) dz|^2$ where $\omega(z)$ is the coefficient of a reference holomorphic differential. This differential has $2g-2$ zeros, which we denote by ξ_I ($I = 1, \dots, 2g-2$). Then, the first factor in the expectation value of (2.9) becomes

$$\frac{Q}{2\pi} \int d^2z \tilde{\phi}(z) \partial \bar{\partial} \log |\omega|^2 = \frac{Q}{2\pi} \int d^2z \tilde{\phi}(z) \sum_{I=1}^{2g-2} (2\pi) \delta^2(z - \xi_I) = Q \sum_{I=1}^{2g-2} \tilde{\phi}(\xi_I), \quad (2.10)$$

where we have used $R = -(2/\sqrt{g})\partial\bar{\partial}\log\sqrt{g}$. Thus, we obtain

$$A_N = \frac{(-\mu)^N}{2bN!} \left\langle \prod_{I=1}^{2g-2} e^{Q\tilde{\phi}(\xi_I)} \int \prod_{i=1}^N d^2 z_i |\omega(z_i)|^2 e^{2b\tilde{\phi}(z_i)} \prod_{k=1}^n e^{-2m_k \tilde{\phi}(w_k)} \right\rangle_{\text{free on } \mathcal{C}_g}. \quad (2.11)$$

The ℓ -point function of the free theory on \mathcal{C}_g is given in the factorized form as [24, 25, 26]

$$\begin{aligned} \left\langle \prod_{i=1}^{\ell} e^{ik_i \phi(z_i, \bar{z}_i)} \right\rangle_{\text{free on } \mathcal{C}} &= (\det \text{Im} \tau)^{1/2} \delta\left(\sum_i k_i\right) \times \\ &\int_{-\infty}^{\infty} \prod_{a=1}^g dp_a \left| \prod_{i=1}^{\ell} \omega(z_i)^{-k_i^2/4} \prod_{i<j} E(z_i, z_j)^{k_i k_j/2} \exp \left(2\pi i \sum_{a,b} p_a p_b \tau_{ab} + 2\pi i \sum_{a,i} p_a k_i \int^{z_i} \omega_a \right) \right|^2, \end{aligned} \quad (2.12)$$

where τ_{ab} is the period matrix, $E(z_i, z_j)$ is the prime form, $\{\omega_a\}$ is a basis of normalized holomorphic one-forms, and p_a is interpreted as the momentum flowing through the a -th A -cycle.

Using the explicit expression (2.12) for (2.9), we find that the residue A_N of the n -point function of the Liouville theory reduces to the following integral

$$\begin{aligned} A_N &\propto \prod_{a=1}^g \int_{-\infty}^{+\infty} dp_a \left| C(w, m, \xi, p) \exp \left(2\pi i \sum_{a,b} p_a p_b \tau_{ab} \right) \right|^2 \\ &\prod_{i=1}^N \int d^2 z_i |\omega(z_i)|^{2+2b^2} \left| \exp \left(4\pi b \sum_{a,i} p_a \int^{z_i} \omega_a \right) \right|^2 \\ &\left| \prod_{i<j} E(z_i, z_j)^{-2b^2} \prod_{i,k} E(z_i, w_k)^{2bm_k} \prod_{I,i} E(\xi_I, z_i)^{-1-b^2} \right|^2, \end{aligned} \quad (2.13)$$

where the factor $C(w, m, \xi, p)$ was defined as

$$\begin{aligned} C(w, m, \xi, p) &= \prod_I \omega(\xi_I)^{\frac{Q^2}{4}} \prod_{k=1}^n \omega(w_k)^{\frac{m_k^2}{g_s^2}} \prod_{k<\ell} E(w_k, w_{\ell})^{-\frac{2m_k m_{\ell}}{g_s^2}} \prod_{I,k} E(\xi_I, w_k)^{\frac{Qm_k}{g_s}} \\ &\times \prod_{I<J} E(\xi_I, \xi_J)^{-\frac{Q^2}{2}} \exp \left[2\pi \sum_a p_a \left(Q \sum_I \int^{\xi_I} \omega_a - 2 \sum_k m_k \int^{w_k} \omega_a \right) \right]. \end{aligned} \quad (2.14)$$

As in the torus case [20], it is not straightforward to factorize the integrals over the Riemann surface into holomorphic and anti-holomorphic integrals for generic N . However this is easily performed in the large N limit. Indeed, the last two-lines of (2.13) can be written as

$$\int \prod_i d^2 z_i |\mu e^{\frac{b}{g_s} W}|^2 \sim \left| \int \prod_i dz_i \mu e^{\frac{b}{g_s} W} \right|^2 \quad (2.15)$$

where μ and W are

$$\mu = \left[\omega(z_i)^{1+b^2} \prod_{i,I} E(z_i, \xi_I)^{-1-b^2} \right] \prod_{1 \leq i < j \leq N} E(z_i, z_j)^{-2b^2} \prod_i E(z_i, z^*)^{2b \sum_k m_k / g_s}, \quad (2.16)$$

$$W = \sum_{i=1}^N \left(\sum_{k=1}^n 2m_k \log \frac{E(z_i, w_k)}{E(z_i, z^*)} + 4\pi \sum_{a=1}^g p_a \int^{z_i} \omega_a \right), \quad (2.17)$$

where we have chosen a base point z^* in order to split the measure from the potential and we have rescaled the parameters as $m_k \rightarrow m_k/g_s$ and $p_a \rightarrow p_a/g_s$. Notice that the term in the square brackets in (2.16) is independent on the zeroes of the conformal factor, ensuring therefore that the generalized matrix model correctly embodies the conformal symmetry of Liouville theory.

The large N limit amounts to take $g_s \rightarrow 0$ keeping $g_s N$, b , m_k and p_a finite. In this limit, the conditions for the criticality are given by

$$b \sum_{j \neq i} \frac{E'(z_i, z_j)}{E(z_i, z_j)} dz_i - \sum_{k=1}^n m_k \frac{E'(z_i, w_k)}{E(z_i, w_k)} dz_i - 2\pi \sum_{a=1}^g p_a \omega_a(z_i) = 0 \quad (2.18)$$

where $E'(z_1, z_2) \equiv \partial_{z_1} E(z_1, z_2)$. The conditions obtained from the \bar{z}_i -derivatives are just the complex conjugate of (2.18). It is remarkable that the conditions for criticality are separated into holomorphic and anti-holomorphic equations, which implies that the integrals over the Riemann surface in (2.13) can be factorized into holomorphic and anti-holomorphic integrals in the large N limit as stated in (2.15). We are therefore left with the following matrix-like integral

$$\begin{aligned} Z_N^{C_{g,n}}(\mathbf{w}, \mathbf{m}, \mathbf{p}, \mathbf{v}) \equiv & \int \prod_{i=1}^N dz_i \left[\omega(z_i)^{1+b^2} \prod_{i,I} E(z_i, \xi_I)^{-1-b^2} \right] \prod_{1 \leq i < j \leq N} E(z_i, z_j)^{-2b^2} \prod_i E(z_i, z^*)^{2b \sum_k m_k / g_s} \\ & \times \exp \left(\frac{b}{g_s} \sum_{i=1}^N \left(\sum_{k=1}^n 2m_k \log \frac{E(z_i, w_k)}{E(z_i, z^*)} + 4\pi \sum_{a=1}^g p_a \int^{z_i} \omega_a \right) \right), \end{aligned} \quad (2.19)$$

where $\mathbf{w} = \{w_k\}$, $\mathbf{m} = \{m_k\}$, $\mathbf{p} = \{p_a\}$ and $\mathbf{v} = \{\nu_\alpha\}$ are the filling fractions $\nu_\alpha \equiv b g_s N_\alpha$ which specify the holomorphic integral above. The integrand in (2.19) is a proper one-form in each variable z_i on the covering space of the Riemann surface due to momentum conservation (2.6). The matrix model potential that we find is in the form anticipated by [13].

In order to count the number of moduli of our matrix model we should note that there are $n + 2g - 3$ independent filling fractions: naively the number of critical points of the action is $2g - 2 + n + 1$. However, there are constraints coming from the fact that we are free to move the base point z^* and that we have specified the residue at the base point as above by using the momentum conservation. The latter is equivalent to the constraint on the sum of filling fractions

$\sum_{\alpha} \nu_{\alpha} = bg_s N$. These constraints reduce the number of moduli by two, thus giving the correct counting. The paths of the integrals are defined such that only the solution of (2.18) labeled by the fixed filling fractions $\{\nu_{\alpha}\}$ contributes to the integrals. The measure factor in (2.19) can be regarded as a generalization of the Vandermonde determinant. The differential $dz\partial_z W$ has simple poles with residues $(\{2m_k\}, -2\sum_k m_k)$ at the points $(\{w_k\}, z^*)$.

The integral in (2.13) is then obtained by integrating (2.19) and its complex conjugate over the filling fractions. Thus, in the large N limit, A_N becomes

$$A_N = \int_{-\infty}^{\infty} \prod_{a=1}^g dp_a \int \prod_{k=1}^{n+2g-3} d\nu_k \left| \exp \left(2\pi i \sum_{a=1}^g \sum_{b=1}^g p_a \tau_{ab} p_b \right) C(\mathbf{w}, \mathbf{m}, \mathbf{p}, \xi) Z_N^{\mathcal{C}_{g,n}}(\mathbf{w}, \mathbf{m}, \mathbf{p}, \mathbf{v}) \right|^2. \quad (2.20)$$

At the level of the generalized matrix model, the filling fractions ν_{α} are free parameters. Together with p_a ($a = 1, \dots, g$) which are independent parameters in the potential, we have totally $n+3g-3$ independent moduli which are identified with the internal momenta in the Liouville conformal block and then with the Coulomb moduli of the gauge theory. The explicit correspondence of these parameters with the internal momenta is discussed in detail in section 4.1. Under this identification, we see from (2.20) that $Z_N^{\mathcal{C}_{g,n}}(\mathbf{w}, \mathbf{m}, \mathbf{p}, \mathbf{v})$ is proportional to the conformal block of Liouville theory [27].

3 Degenerations

To study the behavior of the generalized matrix model when approaching perturbative corners in the space of gauge couplings, we have to study what happens when we degenerate $\mathcal{C}_{g,n}$. The degeneration is usually described by using the plumbing fixture decomposition of the curve. Let \mathcal{U}_t be the annulus

$$\mathcal{U}_t = \{(z, w) | zw = t; |t| < |z| < 1; |t| < |w| < 1\}$$

which as $t \sim 0$ describes the squeezed cylinder. The curve undergoes the decomposition $\mathcal{C}_{g,n} = \mathcal{C}_{g-1,n,2} \cup \mathcal{U}_t$ when the degeneration is of pinching type and $\mathcal{C}_{g,n} = \mathcal{C}_{g_1,n_1,1} \cup \mathcal{U}_t \cup \mathcal{C}_{g_2,n_2,1}$ with $g_1 + g_2 = g$ and $n_1 + n_2 = n$, with $1 - 2g_i - n_i < 0$, when the degeneration is dividing. The components $\mathcal{C}_{g,n,h}$ are here Riemann surfaces with genus g , n punctures and h non overlapping disks removed which will become the punctures in the degeneration limit. The fact that the holomorphic integrals react correctly under the degeneration of the curve $\mathcal{C}_{g,n}$ is a remnant of the analogous property of the conformal field theory [28] and is indeed a consequence of the construction we performed in the previous section. We assume the shrinking cycle do not intersect the contour system along which (2.19) is evaluated.

Let us focus on the dividing case first. In this case the prime form $E(z, z')$ behaves as follows. If both its arguments belong to a given same component, the prime form reduces to the prime form on that component, while if its arguments belong to different components, then $E(z', z'') \sim E_1(z', P_1)E_2(P_2, z'')t^{-1/2}$, where $P_{1,2}$ correspond to the punctures created by the dividing. To see what the prime form degeneration implies for the generalized matrix model measure and potential, we have to split the integration contours in components according to the dividing decomposition. This splits the $\{z_i\}$ in two sets according to which components of the contour they are integrated along, namely N' of them on the first component and N'' on the second with $N = N' + N''$. Correspondingly, also the puncture set will split in two subsets $\mathbf{w} = \mathbf{w}' \cup \mathbf{w}''$, one for each component. By using the above degeneration formulas for the prime form and the fact that the holomorphic harmonic differentials ω_a reduce to the ones relative to the two splitting factors, we get that

$$Z_N^{\mathcal{C}_{g,n}}(\mathbf{w}, \mathbf{m}, \mathbf{p}, \mathbf{v}) \sim Z_{N'}^{\mathcal{C}_{g_1, n_1+1}}(\mathbf{w}' \cup P_1, \mathbf{m}' \cup m_1^*, \mathbf{p}', \mathbf{v}') Z_{N''}^{\mathcal{C}_{g_2, n_2+1}}(\mathbf{w}'' \cup P_2, \mathbf{m}'' \cup m_2^*, \mathbf{p}'', \mathbf{v}'') \quad (3.1)$$

where $m_1^* = bg_s N' - \sum_{k'} m_{k'} + g_s Q(g_1 - 1)$ and $m_2^* = bg_s N'' - \sum_{k''} m_{k''} + g_s Q(g_2 - 1)$ after using momentum conservation. In the computation of (3.1) one needs to count the two extra zeros for the reference holomorphic differential to be placed at the location of the two resulting punctures. The direct computation of the above mass formulas from the integral (2.19) indeed gives, for example, $m_1^* = -bg_s N'' - g_2 g_s Q + \sum_{k''} m_{k''}$ where the first term comes from the generalized Vandermonde, the second from the measure term in the square bracket and the third from the punctures. The computation of m_2^* is identical. Notice that $m_1^* + m_2^* = -g_s Q$ corresponding to the fact that the two Liouville insertions generated at the punctures are conjugated and therefore the masses of the two flavors at the two punctures P_i are equal. The conformal modulus which gets traded to the mass is the total filling fraction between the two integrals. Formula (3.1) is valid for each dividing degeneration such that $g = g_1 + g_2$, $n = n_1 + n_2$ with $1 - 2g_i - n_i < 0$.

In the pinching case, the prime form restricts to the one of the degenerate surface together with the holomorphic differentials with zero α_g -cycle. The left over one scales as $2\pi i \omega_g(z) \sim \partial_z \log \frac{E(z, P_1)}{E(z, P_2)}$ up to $O(t)$ terms. In this limit, since ω in the conformal factor has been chosen to be regular, two of its zeroes will be at the two punctures generated at the pinching node and by direct computation one gets

$$Z_N^{\mathcal{C}_{g,n}}(\mathbf{w}, \mathbf{m}, \mathbf{p}, \mathbf{v}) \sim Z_N^{\mathcal{C}_{g-1, n+2}}(\mathbf{w} \cup \{P_1, P_2\}, \mathbf{m} \cup \{m_+^*, m_-^*\}, \hat{\mathbf{p}}, \mathbf{v}) \quad (3.2)$$

where \hat{p}_a are the momenta in the $g - 1$ left over handles and

$$m_{\pm}^* = -\frac{g_s Q}{2} \pm ip_g, \quad (3.3)$$

with p_g the momentum in the squeezed one. The two contributions to the above mass formulas arise respectively from the term in the square brackets in the measure and the second term in the

potential. Once again, the two masses at the generated punctures are Weyl conjugated $m_+^* + m_-^* = -g_s Q$. In the pinching case the conformal modulus which is traded for the mass is the momentum flowing in the squeezed handle. The formulas above are general and valid at finite N .

In the following subsection we will discuss in detail the punctured torus case as an illustration. One could also study for example the punctured genus two case. This case is special since all degenerations reduce to punctured tori^{*}. We discuss some aspects of the Seiberg-Witten geometry on genus two curves in the Appendix.

3.1 Degenerations of punctured tori

In this subsection, we concentrate on the pinching degeneration of a torus which leads to a sphere with two more punctures. Associated with the torus with n punctures we can consider a class of quiver gauge theories [2, 3] whose particular weak coupling descriptions include a gauge theory with circular quiver. Specifying a particular weak coupling description corresponds to choosing a particular marking (a pants decomposition) of the Riemann surface. This gauge theory will reduce in the pinching of the torus to a linear quiver theory with $n - 1$ $SU(2)$ gauge groups associated with a sphere with $n + 2$ punctures. In what follows, we verify that the generalized matrix model correctly reduces to the Penner type matrix model on the sphere [13].

Since on the torus the canonical bundle is trivial, the choice of a base point is not needed. The prime form is $E(z, w) = \frac{\theta_1(z-w|\tau)}{\theta_1'(0|\tau)}$ and therefore the generalized matrix model (2.19) reduces to

$$Z_N^{\mathcal{C}_{1,n}} \sim \int \prod_{i=1}^N dz_i \prod_{i < j} \theta_1(z_i - z_j)^{-2b^2} e^{\frac{b}{g_s} \sum_i W(z_i)}, \quad (3.4)$$

up to z_i -independent factors. The potential is

$$W(z) = \sum_{k=1}^n 2m_k \log \theta_1(z - w_k) + 4\pi p z, \quad (3.5)$$

and

$$\theta_1(z) = 2 \sin(\pi z) \prod_{m=1}^{\infty} (1 - e^{2\pi i z} q^m)(1 - e^{-2\pi i z} q^m)(1 - q^m), \quad (3.6)$$

^{*}The dividing degeneration $\mathcal{C}_{2,n} \rightarrow \mathcal{C}_{1,n'+1} \cup \mathcal{C}_{1,n-n'+1}$ generates two punctured tori. Indeed, the genus two prime form in such a degeneration reduces to the relevant θ -functions on the two tori since the period matrix at genus two becomes diagonal in the degeneration limit. In the pinching case $\mathcal{C}_{2,n} \rightarrow \mathcal{C}_{1,n+2}$, the genus two θ -function entering the explicit expression of the prime form as $E(z, w) = \frac{\theta(\int_w^z \vec{\omega}, \tau)}{\sqrt{\omega_{\square}(z)} \sqrt{\omega_{\square}(w)}}$ contracts to the torus θ -function times a contribution from the off-diagonal term of the period matrix which cancels in the degeneration limit the square-roots of the abelian differentials appearing in the denominator of the prime form.

with $q = e^{2\pi i\tau}$. The momentum conservation is given by

$$-\sum_{k=1}^n m_k + bg_s N = 0. \quad (3.7)$$

Also, the identification of the moduli of the torus and the gauge coupling constants of $SU(2)$ gauge groups q_k is as follows [5, 20]:

$$e^{2\pi i(w_1-w_2)} = q_1, \quad e^{2\pi i(w_2-w_3)} = q_2, \quad \dots, \quad e^{2\pi i(w_{n-1}-w_n)} = q_{n-1}, \quad e^{2\pi i\tau} = \prod_{k=1}^n q_k, \quad (3.8)$$

which, by fixing $w_n = 0$, leads to

$$e^{2\pi i w_{n-1}} = q_{n-1} \equiv t_{n-1}, \quad e^{2\pi i w_{n-2}} = q_{n-2} q_{n-1} \equiv t_{n-2}, \quad \dots, \quad e^{2\pi i w_1} = q_1 \dots q_{n-1} \equiv t_1. \quad (3.9)$$

The mass parameters and one of the Coulomb moduli correspond to m_k and p respectively.

Let us consider the pinching degeneration of the torus. We take $\text{Im}\tau \rightarrow \infty$ which corresponds in the gauge theory to the decoupling limit of the n -th gauge group $q_n \rightarrow 0$. To consider the behavior of the generalized matrix model in this limit, we first observe that the prime form reduces as

$$\begin{aligned} (dz)^{-1/2} (dw)^{-1/2} \frac{\theta_1(z-w)}{\theta_1'(0)} &\rightarrow (dz)^{-1/2} (dw)^{-1/2} \frac{\sin \pi(z-w)}{\pi} \\ &= (d\xi)^{-1/2} (d\zeta)^{-1/2} (\xi - \zeta), \end{aligned} \quad (3.10)$$

where in the last line we have changed coordinates to $\xi = e^{2\pi i z}$ and $\zeta = e^{2\pi i w}$. It is straightforward to see that the Vandermonde determinant of (3.4) reduces to that of the usual β -deformed matrix model. The potential also reduces to

$$W(\xi) = \sum_{k=1}^n 2m_k \log(\xi - t_k) + 2(-g_s Q/2 - ip) \log \xi, \quad (3.11)$$

where we have used (3.9) with $t_n = 1$. Note that the first term corresponding to the momentum at $\xi = 0$ comes from the measure factor $\omega(z)^{1+b^2}$ of the generalized matrix model (2.19). By the pinching, the punctures at $\xi = 0$ and ∞ are created. However, the latter disappeared from the potential, which thus reduces exactly to the Penner type matrix model [13, 19][†]

$$W(z) = \sum_{k=0}^{n-1} 2m_k \log(z - t_k) + 2m_n \log(z - 1), \quad (3.12)$$

[†]The convention here is slightly different from the one in [13, 18, 19]. The momenta are related as $2m = m_{\text{DV}}$. Our convention leads to the momentum conservation (3.13).

with the momentum conservation

$$-\sum_{k=0}^n m_k - m_\infty + bg_s N = g_s Q, \quad (3.13)$$

where m_∞ is the momentum inserted at infinity. The relation between the parameters t_k and the gauge couplings [19] (See also [7]) is the same as the one defined in (3.9) with $t_0 = 0$. It follows from (3.11) that $m_0 = -g_s Q/2 - ip$.

Let us then analyze the momentum conservation under this degeneration. On one hand, in the original generalized matrix model, the conservation is described by (3.7). On the other hand, in the Penner type one, the conservation is (3.13). The momentum at infinity is then

$$m_\infty = -\frac{g_s Q}{2} + ip. \quad (3.14)$$

These values of the momenta m_0 and m_∞ are the ones which were already derived in the generic analysis of the previous section (3.3). Note that there is a slight difference between the momenta m_0 and m_∞ , which however disappears in the large N limit.

The original generalized matrix model has $n - 1$ independent filling fractions ν_α . Recall that the overall $\sum_\alpha \nu_\alpha = bg_s N$ is constrained by the momentum conservation. Thus, by adding p , we have n independent parameters which are identified with the vevs of the vector multiplet scalars. The degeneration limit and the above argument mean that p in the potential is the vev of the n -th $SU(2)$ vector multiplet scalar and some combinations of the filling fractions are the vevs of the other $SU(2)$ scalars. We will give the precise identification in the large N limit in the next section.

4 Spectral curve of the generalized matrix model

In this section, we derive the spectral curve of the generalized matrix model (2.19) in the large N limit and show that it coincides with the Seiberg-Witten curve of the corresponding gauge theory.

In the large N limit, the evaluation of (2.19) reduces to the calculation of the critical points. The condition for criticality is given by

$$dW(z_i) - 2bg_s \sum_{j \neq i} d_{z_i} \log \left(\frac{E(z_i, z_j)}{E(z_i, z_*)} \right) = 0, \quad (4.1)$$

where the potential $W(z)$ is defined in (2.17) and we have used the momentum conservation (2.6). Then, the prepotential in the large N limit, defined as $\exp(\mathcal{F}/g_s^2) \equiv Z$, is given by

$$\frac{1}{g_s^2} \mathcal{F} = \frac{b}{g_s} \sum_i W(z_i) - 2b^2 \sum_{i < j} \log \left(\frac{E(z_i, z_j)}{E(z_i, z_*)} \right), \quad (4.2)$$

where each “eigenvalue” z_i satisfies (4.1).

It is natural to assume that the eigenvalues are distributed in line segments around the critical points of $W(z)$, similarly to the usual matrix model. Indeed the second term in (4.2) reduces locally to the standard Coulomb gas potential. We denote the line segments as C_α where $\alpha = 1, \dots, n + 2g - 2$. We assume that C_α do not include the base point z^* and the punctures w_k , at which the potential $W(z)$ diverges. We denote by N_α the number of eigenvalues on the line segment C_α , where N_α satisfies $\sum_{\alpha=1}^{n+2g-2} N_\alpha = N$.

Let us introduce the eigenvalue density current $\rho(z)$ supported on $\{C_\alpha\}$ and normalized as $\oint_{C_\alpha} \rho(z) = bg_s N_\alpha \equiv \nu_\alpha$. Using the variables introduced above, the prepotential and the condition for criticality are written as

$$\mathcal{F} = \int_{\sum_\alpha C_\alpha} \rho(z) W(z) - \int_{\sum_\alpha C_\alpha} \int_{\sum_\alpha C_\alpha} \rho(z) \rho(z') \log \frac{E(z, z')}{E(z, z^*)}, \quad (4.3)$$

$$dW(z) - 2 \int_{\sum_\alpha C_\alpha} \rho(z') dz \log \left(\frac{E(z, z')}{E(z, z^*)} \right) = 0, \quad (4.4)$$

respectively. Here, z in (4.4) is on either of the line segment C_α and the integral is defined as the principal integral.

In order to solve the above condition (4.4), we define the following one form, which is the generalization of the resolvent of the usual matrix model

$$R(z) \equiv \int_{\sum_\alpha C_\alpha} \rho(z') dz \log \left(\frac{E(z, z')}{E(z, z^*)} \right). \quad (4.5)$$

This “resolvent” is defined at generic points z on the Riemann surface contrary to the second term in (4.4). Note that the resolvent as well as $dW(z)$ are single-valued one-forms on the Riemann surface. The resolvent has cuts at the line segments C_α and a simple pole at z^* . Also, the filling fractions are obtained by integrating the resolvent along the cuts as

$$\nu_\alpha = \frac{1}{2\pi i} \oint_{C_\alpha} R(z). \quad (4.6)$$

On the line segments C_α , the resolvent behaves as

$$R(z + i\varepsilon e^{i\varphi(z)}) + R(z - i\varepsilon e^{i\varphi(z)}) = 2P \int_{\sum_\alpha C_\alpha} \rho(z') dz \log \left(\frac{E(z, z')}{E(z, z^*)} \right) = dW(z), \quad (4.7)$$

$$R(z + i\varepsilon e^{i\varphi(z)}) - R(z - i\varepsilon e^{i\varphi(z)}) = \oint_z \rho(z') dz \log \left(\frac{E(z, z')}{E(z, z^*)} \right) = -2\pi i \rho(z) \quad (4.8)$$

where we take the real number ε infinitely small and we assume that a properly defined function $\varphi(z)$ exists such that $z + i\varepsilon e^{i\varphi(z)}$ or $z - i\varepsilon e^{i\varphi(z)}$ does not go across the cuts C_α when z moves along C_α . The integral in (4.7) is principal integration, which is given as an average of integral along the

path above the singularity and that below the singularity. The resolvent should be determined such that (4.7) and (4.8) are satisfied for $z \in C_\alpha$. A candidate of the solution for (4.7) is

$$R_0(z) = \frac{1}{2}dW(z). \quad (4.9)$$

However, it does not reproduce the correct structure of singularity expressed in (4.8). We need singular contributions:

$$R(z) = \frac{1}{2}dW(z) + R(z)_{\text{sing}}, \quad (4.10)$$

where (4.7) and (4.8) impose

$$R(z + i\varepsilon e^{i\varphi(z)})_{\text{sing}} + R(z - i\varepsilon e^{i\varphi(z)})_{\text{sing}} = 0. \quad (4.11)$$

$$R(z + i\varepsilon e^{i\varphi(z)})_{\text{sing}} - R(z - i\varepsilon e^{i\varphi(z)})_{\text{sing}} = -2\pi i \rho(z). \quad (4.12)$$

The above discussion is valid for a generic potential $W(z)$. In the following, we use its explicit form (2.17) to determine the resolvent $R(z)$. Then, we find that

$$\begin{aligned} R_{\text{sing}}(z) &= \int_{\sum_\alpha C_\alpha} \rho(z') d_z \log \left(\frac{E(z, z')}{E(z, z^*)} \right) - \sum_{k=1}^n m_k d_z \log \left(\frac{E(z, w_k)}{E(z, z^*)} \right) - 2\pi \sum_{a=1}^g p_a \omega_a(z) \\ &= \int_{\sum_\alpha C_\alpha} \rho(z') d_z \log E(z, z') - \sum_{k=1}^n m_k d_z \log E(z, w_k) - 2\pi \sum_{a=1}^g p_a \omega_a(z) \end{aligned} \quad (4.13)$$

does not depend on the base point z^* , where we used the momentum conservation (2.6) and ignored the subleading term in the large N expansion. We see that $R(z)_{\text{sing}}$ has cuts in the regions C_α and simple poles with residues m_k at $z = w_k$. Moreover, it is independent on the base point z^* as expected.

From (4.11), we see that the sign of $R(z)_{\text{sing}}$ changes across the cuts. Therefore its square only displays singularities at the punctures $z = w_k$. From (4.13) we see that these are at most quadratic poles with coefficients m_k^2 . The spectral curve of the generalized matrix model thus reads

$$R_{\text{sing}}(z)^2 = \sum_{k=1}^n m_k^2 \eta(z, w_k) + \zeta(z) \quad (4.14)$$

where $\eta(z, w_k)$ are quadratic Strebel differentials, with double pole at w_k , and $\zeta(z)$ is a quadratic differential which has at most simple poles at w_k . $\zeta(z)$ is determined in terms of $n + 3g - 3$ parameters; in particular it depends on the $n + 2g - 3$ independent filling fractions ν_α and the g internal momenta p_a . This form of the curve is the same as that of the Seiberg-Witten curve of quiver gauge theory as a cover of the base Riemann surface [2, 3]. As discussed in [29], the physical information is included in the Prym variety of the Seiberg-Witten curve, rather than the Jacobian variety. This reduces the number of independent periods of the Seiberg-Witten differential to $n + 3g - 3$, which agrees with the number of the filling fractions and p_a .

In order for the above argument to be a check of AGT correspondence at all genera, we need to show that the spectral curve is indeed the same as the one proposed, in the Virasoro conformal block side, to be the Seiberg-Witten curve: the insertion of the energy-momentum tensor in the conformal block

$$\left\langle T(z) \prod_{k=1}^n V_{\alpha_k}(w_k) \right\rangle_{C_g} \rightarrow -\frac{x(z)^2}{g_s^2} \left\langle \prod_{k=1}^n V_{\alpha_k}(w_k) \right\rangle_{C_g} \quad (4.15)$$

in the semi-classical limit, which corresponds to the large N limit in the generalized matrix model. It was claimed that xdz coincides with the Seiberg-Witten differential. This was checked in the cases on a sphere and a torus already in [5]. A useful way to capture the energy-momentum tensor insertion is to consider the insertion of the degenerate field $\Phi_{1,2}(z) \equiv e^{-\frac{\phi(z)}{b}}$ in the conformal block. By this insertion, the conformal block satisfies the BPZ equation and can be expanded in g_s as [9]

$$\left\langle \Phi_{1,2}(z) \prod_{k=1}^n V_{\alpha_k}(w_k) \right\rangle_{C_g} \sim \exp \left(\frac{\mathcal{F}_0}{g_s^2} + \frac{1}{bg_s} \int^z x(z') dz' + \dots \right) \quad (4.16)$$

Thus, the counterpart of the Seiberg-Witten differential in the Virasoro side can be obtained by calculating

$$x(z) = bg_s \frac{\partial}{\partial z} \log \frac{\langle \Phi_{1,2}(z) \prod_{k=1}^n V_{\alpha_k}(w_k) \rangle_{C_g}}{\langle \prod_{k=1}^n V_{\alpha_k}(w_k) \rangle_{C_g}} \quad (4.17)$$

in the semi-classical limit.

In the following, we rewrite (4.17) in terms of the generalized matrix model by using the discussion in section 2. The corresponding calculation in the case of the sphere has been done in [21]. We use the holomorphic half of the integrand in (2.20) to evaluate the conformal block in (4.17). The conformal block with the insertion is also obtained just by changing n to $n+1$ and by regarding that $m_{n+1} = \frac{g_s}{2b}$ and $w_{n+1} = z$. The momentum conservation (2.6) is slightly modified to

$$\sum_{k=1}^n m_k + \frac{g_s}{2b} + g_s Q(1-g) = bg_s N. \quad (4.18)$$

By collecting the factor dependent on z , we obtain

$$\begin{aligned} x(z) &= bg_s \frac{\partial}{\partial z} \log \left(\omega(z)^{\frac{1}{4b^2}} \prod_I E(z, \xi_I)^{\frac{Q}{2b}} E(z, z^*)^{N - \frac{\sum_k m_k}{bg_s}} e^{-\frac{W(z)}{2bg_s}} \left\langle \prod_i \frac{E(z, z_i)}{E(z, z^*)} \right\rangle \right) \\ &= -\frac{1}{2} \frac{\partial W(z)}{\partial z} + bg_s \frac{\partial}{\partial z} \log \left\langle \prod_i \frac{E(z, z_i)}{E(z, z^*)} \right\rangle + \mathcal{O}(N^{-1}), \end{aligned} \quad (4.19)$$

where we have used the deformed momentum conservation (4.18) in the second equality. The expectation value is defined by

$$\langle \dots \rangle = \frac{\int \prod dz_i \mu e^{\frac{b}{g_s} \sum_i W(z_i)} \dots}{\int \prod dz_i \mu e^{\frac{b}{g_s} \sum_i W(z_i)}}. \quad (4.20)$$

A subtlety is that the numerator in (4.20) is defined with the deformed momentum conservation (4.18) while the denominator with the original momentum conservation. However, note that the effect of the insertion of the degenerate field is a subleading contribution in the large N limit, as in (4.16). Note also that the deformation of the momentum conservation gives rise to that of the prepotential \mathcal{F}_0 , which does not depend on z and disappears by the partial derivative in terms of z . Thus, we can use the same external momenta m_k and the same N as those of the denominator of (4.20) to evaluate the numerator in the expectation value. Also, some of the internal momenta in the numerator are shifted by $\pm b/2$ depending on where we insert the degenerate operator as discussed in [9], but again, the effect of this shift does not produce the factor dependent on z in the large N limit. Thus, the expectation value in (4.19) can be evaluated by substituting the solution of the condition of criticality (4.1) or (4.4), which leads to

$$bg_s \frac{\partial}{\partial z} \log \left\langle \prod_i \frac{E(z, z_i)}{E(z, z^*)} \right\rangle = bg_s \left\langle \sum_i \frac{\partial}{\partial z} \log \frac{E(z, z_i)}{E(z, z^*)} \right\rangle + \mathcal{O}(N^{-1}). \quad (4.21)$$

From the discussion above, we finally obtain

$$x(z)dz \sim -\frac{dW(z)}{2} + R(z) = R_{\text{sing}}(z), \quad (4.22)$$

up to $\mathcal{O}(N^{-1})$ terms. This shows that the resolvent R_{sing} is indeed $x(z)dz$ in (4.15), which was claimed to agree with the Seiberg-Witten differential.

4.1 Dependence on internal momenta

In this subsection, we show that the spectral curve of the generalized matrix model depends on the Coulomb moduli parameters in the same way as the Seiberg-Witten curve, which completes the check of AGT correspondence at all genera. In order to do that, we first have to specify the marking of the Riemann surface $\mathcal{C}_{g,n}$, which is done by choosing $n+3g-3$ physically independent cycles. This marking determines the conformal block labeled by a trivalent graph on one hand, and also the corresponding weak coupling description of the gauge theory on the other hand. On the gauge theory side, each vacuum expectation value of each vector multiplet scalar is obtained by an integral along each cycle of the marking γ_r :

$$a_r = \frac{1}{2\pi i} \oint_{\gamma_r} \lambda_{\text{SW}}, \quad (4.23)$$

where λ_{SW} is the Seiberg-Witten differential. We would like to show that the spectral curve also satisfies the corresponding relation.

Before going to that issue, we show in the following discussion that each pair of pants has one cut under the marking. In general, three-punctured sphere always has one cut [15] because the classical potential of the corresponding Penner type matrix model with generic mass parameters has one extremum. We can also show this by considering the dividing limit of four-punctured sphere: under the decomposition, it is known that one cut exists in the pants including the punctures originally at 0 and q , while the other cut exist in the pants including the punctures at 1 and ∞ . Furthermore, unless the mass parameters vanish, there are no punctures belonging to the cut. This indicates that three-punctured sphere with generic mass parameters has one cut apart from the three punctures. Since we have shown that our generalized matrix model behaves correctly in the degeneration limit, the cuts of its resolvent should be placed in such a way that each pair of pants has one cut.

Once the marking is specified, an explicit correspondence should be determined between the $n + 3g - 3$ internal momenta of the Virasoro conformal block and the $n + 2g - 3$ independent filling fractions together with the n parameters p_a in the potential. In order to see this explicitly, we can use the results on the degeneration obtained in section 3. Note that among the intermediate states in the conformal block, only the primary state remains and appears as the external state in the degeneration limit. Thus, the external momentum appeared by the degeneration which shrinks a particular cycle can be seen as the internal momentum of the corresponding place before the degeneration. In order to determine all the internal momenta, we pinch all the A -cycles of the Riemann surface $\mathcal{C}_{g,n}$ until it becomes $\mathcal{C}_{0,n+2g}$ and then divide $\mathcal{C}_{0,n+2g}$ until they split into three-punctured spheres in a way that it reproduces the specified pants decomposition. Whichever marking we take, we can read off all the internal momenta by considering the corresponding degeneration limit.

The A -cycles of $\mathcal{C}_{g,n}$ can be chosen to coincide with some of γ_r 's. Thus, the set $\{\gamma_r\}$ can be divided as $\{\gamma_r\} = \{A_a\} \cup \{\gamma_\alpha\}$ where $a = 1, \dots, g$ and $\alpha = 1, \dots, n + 2g - 3$, such that A -cycles of $\mathcal{C}_{g,n}$ are A_a and γ_α are the remaining cycles. By shrinking A_a -cycles, we obtain (3.3). Especially, in the large N limit, the internal momenta (multiplied by g_s) corresponding to the A_a cycles of the Riemann surface \mathcal{C}_g are given by

$$a_a = ip_a. \quad (a = 1, \dots, g) \quad (4.24)$$

After pinching all the A_a cycles, the remaining internal momenta can be obtained as written just below (3.1) by further considering the dividing limit which shrinks γ_α cycles. Since each pair of pants has one cut, on which N_β eigenvalues $\{z_i\}$ exist, the number of eigenvalues N' or N'' on each side of the dividing component of the Riemann surface can be written in terms of sum of the filling fractions. In the large N limit, the internal momenta (multiplied by g_s) flowing through γ_α

cycles are given by

$$a_\alpha = \sum_{\beta} \nu_\beta - \sum_k m_k + \sum_a (\pm i p_a), \quad (\alpha = 1, \dots, n + 2g - 3) \quad (4.25)$$

where the range of the sum depends which γ_α cycle we shrink under the marking of the Riemann surface. The third term is the contribution from new punctures which appear by pinching A_a cycles in the previous step. The sign \pm is determined for each puncture according to the direction of the A_a cycle. Although the internal momenta (4.24) and (4.25) are obtained in the degeneration limit, we assume that these relations hold for arbitrary moduli parameters of the Riemann surface $\mathcal{C}_{g,n}$ because the filling fraction ν_α and the parameter p_a are independent of the moduli parameters. It is remarkable that the form of our generalized matrix model in the large N limit is universal for any choice of markings of the Riemann surface. However, the identification of the parameters with internal momenta depend on such choice, which reflects in the difference of the conformal blocks labeled by different trivalent graphs.

So far, we have discussed the relation between the internal momenta of the Virasoro conformal block and the independent filling fractions and parameters p_a . In the following, we check that the corresponding relation as (4.23) is reproduced from the spectral curve under these identification of the parameters. By using (4.13), we can explicitly calculate the A_a -cycle integrals as

$$\frac{1}{2\pi i} \int_{A_a} R_{\text{sing}} = i p_a \quad (a = 1, \dots, g) \quad (4.26)$$

Note that the convention here is $\oint_{A_a} \omega_b = \delta_b^a$. Taking into account the identification of the parameters (4.24), we see that the corresponding relation as (4.23) for A_a -cycle is confirmed.

For the γ_α cycles, the check is not direct, because γ_α are not the contours C_α along the cuts of the spectral curve of the matrix model. Recall that there are indeed $n + 2g - 2$ cuts in the spectral curve, but the integral of the resolvent along one of them is not independent due to the momentum conservation. We will call the remaining $n + 2g - 3$ cuts the independent ones. It follows that the cycles around the independent cuts, which we have denoted by C_α ($\alpha = 1, \dots, n + 2g - 3$), can be expressed as sum of the three cycles which go around each leg of the pants. That is, C_α can be written as a linear combination of γ_r ($r = 1, \dots, n + 3g - 3$) and D_k ($k = 1, \dots, n$), where D_k are the contours around the simple poles of R_{sing} . Thus, γ_α ($\alpha = 1, \dots, n + 2g - 3$) can be written as $\gamma_\alpha = n_\alpha^a A_a + c_\alpha^\beta C_\beta + s_\alpha^k D_k$, where the coefficients n_α^a , c_α^β and s_α^k are integers. (Note that in addition to them there could exist a cycle around the base point. However, when we consider R_{sing} , this dependence disappears in the large N limit, as stated in (4.13).) In terms of these, we

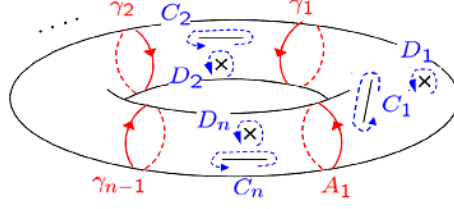


Figure 1: A marking of $\mathcal{C}_{1,n}$ corresponding to the elliptic quiver and the A_1 - and γ_α -cycles. E.g., $\gamma_1 = A_1 + C_1 + D_1$ and $\gamma_2 = A_1 + C_1 + C_2 + D_1 + D_2$.

calculate the integral over γ_α cycles as

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma_\alpha} R_{\text{sing}}(z) &= \frac{1}{2\pi i} \oint_{\gamma_\alpha} \left[R(z) - \sum_{k=1}^n m_k \omega_{w_k, z^*}(z) - 2\pi \sum_{a=1}^g p_a \omega_a(z) \right] \\ &= \sum_{\beta} c_{\alpha}^{\beta} \nu_{\beta} - \sum_k s_{\alpha}^k m_k + i \sum_a n_{\alpha}^a p_a, \end{aligned} \quad (4.27)$$

where $\omega_{x,y}(z) = d_z \log \frac{E(z,x)}{E(z,y)}$ is the Abelian differential of the third kind.

To determine the coefficients, we briefly show the example below. In the case of the torus $\mathcal{C}_{1,n}$, there is only one A -cycle and the number of independent cuts is $n - 1$. For $n = 1$, there is no independent cut and the parameter p is identified with the vacuum expectation value of the vector multiplet scalar of $\mathcal{N} = 2^*$ gauge theory as analyzed in [20]. For $n > 1$, we consider the marking corresponding to $SU(2)^n$ elliptic quiver gauge theory and choose the A_1 -cycle and γ_α -cycles as depicted in Fig 1. As stated above, each pair of pants has one cut. (This can also be explicitly checked by solving the equations of motion in the weak coupling limit.) Therefore, it is easy to obtain the coefficients above $n_{\alpha}^1 = 1$ and

$$c_{\alpha}^{\beta} = \begin{cases} 1 & (\beta = 1, \dots, \alpha) \\ 0 & (\beta = \alpha + 1, \dots, n - 1) \end{cases} \quad s_{\alpha}^k = \begin{cases} 1 & (k = 1, \dots, \alpha) \\ 0 & (k = \alpha + 1, \dots, n - 1) \end{cases} \quad (4.28)$$

This indicates

$$\frac{1}{2\pi i} \oint_{\gamma_\alpha} R_{\text{sing}}(z) = \sum_{\beta=1}^{\alpha} \nu_{\beta} - \sum_{k=1}^{\alpha} m_k + ip \quad (4.29)$$

in addition to the one obtained by the A -cycle integral (4.26). Comparing it with the result obtained from (4.25), we see that it exactly reproduces the internal momenta flowing through γ_α cycle. Similarly, it is straightforward to check for generic case that (4.27) coincides with that obtained from (4.25). Thus, we have confirmed that the corresponding relations as (4.23) for γ_α cycles are reproduced.

5 Conclusions

In this paper, we have shown that the perturbative analysis of Liouville correlation functions displays in the large N limit holomorphic factorization of the surface integrals and leads to generalized matrix models, defined on the cover of $\mathcal{C}_{g,n}$, which describe the relevant Virasoro conformal blocks. We provided an all genera check of the AGT correspondence by obtaining the Seiberg-Witten data from the saddle point analysis of these generalized matrix models.

We underline that the models presented in this paper could be useful for the exploration of the full set of gauge theories with generalized quiver structure of [3]. Indeed, so far most of the analysis of the AGT correspondence has been focused on the linear and elliptic quiver cases, mainly due to the lack of calculational tools for higher genera. However, to fully exploit the generalized matrix model approach one should be able to extend its analysis to finite N . This would amount to provide a full derivation at finite N of the holomorphic factorization, which in turn would give a precise prescription for the contour integrals possibly extending the recipe of [16, 22] to higher genera.

Notice anyway that our large N analysis depends only on the homotopy class of the contours via the filling fractions as explained in Section 4.1. In this sense our results are universal with respect to a particular choice of contour's representatives in the matrix integral.

Moreover, for gauge groups of higher rank, which according to [13, 14] should correspond to multi-matrix models, this approach could shed light on the description of strongly coupled sectors naturally appearing in the general framework and not admitting a known lagrangian description.

Another very interesting issue to explore is the relation of the generalized matrix models with the quantization of integrable systems [30, 11]. (See also [31].) In particular this could provide an alternative derivation of the quantum Hamiltonians for Hitchin integrable systems and generalize it to higher genera.

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Appendix

A Addendum: playing with genus 2 curves

A hyperelliptic curve \mathcal{C}_g of genus g is given by the equation

$$y^2 = P_{2g+2}(x)$$

where P_{2g+2} is a polynomial of degree $2g + 2$ and is realized ‡ in the total space of $T\mathbb{P}^{\frac{g+1}{2}}$.

As it is well known, all genus 2 curves are hyperelliptic. These are realized in general by a sextic polynomial equation

$$y^2 = \prod_{i=1}^6 (x - a_i) \tag{A.1}$$

which we denote by \mathcal{C}_2 .

The complex structure moduli $\overline{\mathcal{M}}_{2,0}$ of genus two curves is then obtained by considering the complex parameters $\{a_i\}$ modulo the action of the permutation group S_6 and the $PSL(2, \mathbb{C})$.

A basis of abelian differentials is given by $\omega_a = \frac{x^{a-1}dx}{y}$, $a = 1, 2$, while a basis of quadratic differentials is given by $\phi_\alpha = \frac{x^{\alpha-1}dx^2}{y^2}$, $\alpha = 1, 2, 3$ [32].

The Seiberg-Witten (SW) geometry of the $SU(2)$ theory at genus 2 is specified by a double cover of \mathcal{C}_2 in $T^*\mathcal{C}_2$. As such, this is specified by a general quadratic differential on \mathcal{C}_2 in the form

$$w^2 = \Phi_2 \tag{A.2}$$

where $\Phi_2 = \sum_\alpha K_\alpha \phi_\alpha$ can be expanded in the Coulomb moduli K_α of the theory.

The perturbative expansions of the theory are available in the vicinity of the degeneration locus of the moduli space, namely around

$$\partial\overline{\mathcal{M}}_{2,0} = \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1} \cup \overline{\mathcal{M}}_{1,2} \tag{A.3}$$

The second factor in (A.3) is still generically not lagrangian and has to be degenerated as $\partial\overline{\mathcal{M}}_{1,2} = \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,3} \cup \overline{\mathcal{M}}_{0,4}$ to reach corners around which known lagrangian descriptions are available. The first factor in (A.3), being given by two copies of the $\mathcal{N} = 2^* SU(2)$ theory, is already lagrangian. The first degeneration is dividing and the second one is pinching.

Let us discuss the dividing case in detail. This is reached by taking the limit in which three branch points in (A.1) collide. To be concrete, let's fix the position of two of them at 0 and ∞ ,

‡ Under conformal inversion $x = 1/x'$ on the Riemann sphere, the stability of the description is guaranteed by the transformation $y = y'x'^{-(g+1)} = y' \left(\frac{\partial x}{\partial x'} \right)^{\frac{g+1}{2}}$.

write our curve as

$$y^2 = x(x - a_1\epsilon)(x - a_2\epsilon)(x - a_3)(x - a_4) \quad (\text{A.4})$$

and take the limit as $\epsilon \rightarrow 0$. The curve (A.4) becomes in the x coordinate

$$y^2 = x^3(x - a_3)(x - a_4)$$

which, redefining $y = x\tilde{y}$, reads

$$\tilde{y}^2 = x(x - a_3)(x - a_4) \quad (\text{A.5})$$

that is the torus with a puncture at $x = 0$.

Let us check now the SW geometry in the degeneration limit. The issue to discuss is just the scaling of the Coulomb parameters in this limit. Notice that the degeneration is obtained by contracting to zero two nearby branching points, therefore we are saturating a complex structure modulus corresponding to a Beltrami differential μ_ϵ with support around the origin of size $\sim \epsilon$. This is dual to a holomorphic quadratic differential which, not to have a vanishing overlap integral with μ_ϵ should not be zero at $x = 0$. This is uniquely determined to be $\frac{dx^2}{y^2}$. Therefore, along the limit with $\epsilon \rightarrow 0$, the corresponding parameter in the SW curve has to scale away.

As a consequence of the above reasoning, exposing the ϵ -parameter, the SW curve is parametrized as

$$w^2 = \frac{u'\epsilon + m^2x + ux^2}{y^2}(dx)^2 \quad (\text{A.6})$$

The degeneration of \mathcal{C}_2 is easily kept into account in the SW geometry which becomes

$$w^2 = \frac{m^2x^{-1} + u}{\tilde{y}^2}(dx)^2 \quad (\text{A.7})$$

The standard parameterization of the punctured torus is in the coordinates where the puncture sits at ∞ . Therefore, we rewrite the elliptic curve (A.5) after the inversion $x = \frac{1}{x'}$ to pull the puncture at $x' = \infty$ and redefine accordingly $\tilde{y} = \frac{1}{(x')^2}\tilde{y}'$. After this, the SW curve reads

$$w^2 = \frac{m^2x' + u}{(\tilde{y}')^2}(dx')^2$$

which we can put in the representation with respect to the periodic coordinate via the Weierstrass parameterization[§] $x' = \mathcal{P}(z) + c$ and $\tilde{y}' = \frac{dz}{dz}\mathcal{P}(z)$ so that we stay with

$$w^2 = [m^2\mathcal{P}(z) + (u + cm^2)](dz)^2 \quad (\text{A.8})$$

which is the SW curve for a copy of the $\mathcal{N} = 2^*$ theory.

[§]The constant c needs to bring (A.5) to the standard Weierstrass form where the quadratic term vanishes and can be computed explicitly.

The other copy corresponds to the other half in which the original genus 2 surface was split. Let's see how to get this second copy. In order to do it we have to consider the curve in the coordinate appropriate for the other half, namely we have to change (A.4) to $x = \epsilon/\hat{x}$ and correspondingly $y = \hat{y} \frac{\epsilon^{3/2}}{(\hat{x})^3}$ after which we get

$$\hat{y}^2 = \hat{x}(1 - a_1\hat{x})(1 - a_2\hat{x})(\epsilon - a_3\hat{x})(\epsilon - a_4\hat{x}). \quad (\text{A.9})$$

(A.9) becomes after the degeneration limit the curve

$$\hat{y}^2 = \hat{x}^3(1 - a_1\hat{x})(1 - a_2\hat{x})(a_3a_4)$$

which we bring to the form of a punctured torus by redefining $\hat{y} = \tilde{y}\hat{x}$ and get

$$\tilde{y}^2 = \hat{x}(1 - a_1\hat{x})(1 - a_2\hat{x})(a_3a_4) .$$

Let's follow now what happens to the SW curve (A.6) in the \hat{x} patch. This becomes

$$w^2 = \frac{u' + m^2\hat{x}^{-1} + u\epsilon\hat{x}^{-2}}{\hat{y}^2}(d\hat{x})^2 \quad (\text{A.10})$$

which in the limit $\epsilon \rightarrow 0$ has the same form of (A.7), but referring to the second punctured torus with an independent Coulomb parameter u' . So, following the same procedure leading to (A.8), we get the second copy of $\mathcal{N} = 2^*$ with an independent gauge coupling and Coulomb parameter but the same mass as the first.

A pinching of the genus 2 curves (A.4) can be obtained by letting $\epsilon \rightarrow \frac{a_3}{a_2}$ for example. In such a case the curve gets to

$$y^2 = x(x - a_1a_3/a_2)(x - a_3)^2(x - a_4) \quad (\text{A.11})$$

which is, after renaming $y = \tilde{y}(x - a_3)$, the twice punctured torus

$$\tilde{y}^2 = x(x - a_1a_3/a_2)(x - a_4). \quad (\text{A.12})$$

Correspondingly, the holomorphic quadratic differential entering the Seiberg-Witten curve (A.6) becomes

$$\Phi_2 \rightarrow \frac{u'a_3/a_2 + m^2x + ux^2}{(x - a_3)^2\tilde{y}^2}(dx)^2 \quad (\text{A.13})$$

which explicitly displays quadratic poles at the two images of $x = a_3$ with equal coefficients. This coefficient is actually fixed by the Coulomb parameter corresponding to the ungauged group $SU(2)$ at the end of the shrinking of the handle.

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